The two dimensional harmonic oscillator on a noncommutative space with minimal uncertainties

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Abstract: The two dimensional set of canonical relations giving rise to minimal uncertainties previously constructed from a q-deformed oscillator algebra is further investigated. We provide a representation for this algebra in terms of a flat noncommutative space and employ it to study the eigenvalue spectrum for the harmonic oscillator on this space. The perturbative expression for the eigenenergy indicates that the model might possess an exceptional point at which the spectrum becomes complex and its PT-symmetry is spontaneously broken.

In [1] we demonstrated how canonical relations implying minimal uncertainties can be derived from a q-deformed oscillator algebra for the creation and annihilation operators $A_i^\dagger$, A_i

$$A_i A_j^\dagger - q^{2\delta_{ij}} A_j^\dagger A_i = \delta_{ij}, \quad [A_i^\dagger, A_j^\dagger] = 0, \quad [A_i, A_j] = 0, \qquad \text{for } i,j = 1,2,3; q \in \mathbb{R}, \tag{1}$$

as investigated for instance in [2, 3, 4, 5, 6]. Starting from the general Ansatz

$$X = \hat{\kappa}_1(A_1^\dagger + A_1) + \hat{\kappa}_2(A_2^\dagger + A_2) + \hat{\kappa}_3(A_3^\dagger + A_3), \tag{2}$$
$$Y = i\hat{\kappa}_4(A_1^\dagger - A_1) + i\hat{\kappa}_5(A_2^\dagger - A_2) + i\hat{\kappa}_6(A_3^\dagger - A_3), \tag{3}$$
$$Z = \hat{\kappa}_7(A_1^\dagger + A_1) + \hat{\kappa}_8(A_2^\dagger + A_2) + \hat{\kappa}_9(A_3^\dagger + A_3), \tag{4}$$
$$P_x = i\check{\kappa}_{10}(A_1^\dagger - A_1) + i\check{\kappa}_{11}(A_2^\dagger - A_2) + i\check{\kappa}_{12}(A_3^\dagger - A_3), \tag{5}$$
$$P_y = \check{\kappa}_{13}(A_1^\dagger + A_1) + \check{\kappa}_{14}(A_2^\dagger + A_2) + \check{\kappa}_{15}(A_3^\dagger + A_3), \tag{6}$$
$$P_z = i\check{\kappa}_{16}(A_1^\dagger - A_1) + i\check{\kappa}_{17}(A_2^\dagger - A_2) + i\check{\kappa}_{18}(A_3^\dagger - A_3), \tag{7}$$

with $\hat{\kappa}_i = \kappa_i\sqrt{\hbar/(m\omega)}$ for $i = 1,\ldots,9$ and $\check{\kappa}_i = \kappa_i\sqrt{m\omega\hbar}$ for $i = 10,\ldots,18$ we constructed some particular solutions and investigated the harmonic oscillator on these spaces. Here we provide an additional two dimensional solution previously reported in [6]. Setting $\kappa_3 = \kappa_6 = \kappa_7 = \kappa_{12} = \kappa_{15} = \kappa_{16} = \kappa_{17} = \kappa_{18} = 0$ in equations (2)-(7), employing the constraints reported in [6] together with the subsequent nontrivial limit $q \to 1$, the deformed oscillator algebra

$$\begin{array}{lll} [X,Y] = i\theta\left(1+\hat{\tau}Y^2\right), & [X,P_x] = i\hbar\left(1+\hat{\tau}Y^2\right), & [X,P_y] = 0, \\ [P_x,P_y] = i\hat{\tau}\frac{\hbar^2}{\theta}Y^2, & [Y,P_y] = i\hbar\left(1+\hat{\tau}Y^2\right), & [Y,P_x] = 0, \end{array} \tag{8}$$

was obtained, with $\hat{\tau} = \tau m\omega/\hbar$ having the dimension of an inverse squared length. By the same reasoning as provided in [7, 8, 5, 9, 6, 1], we find the minimal uncertainties

$$\Delta X_{\min} = |\theta| \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2}, \quad \Delta Y_{\min} = 0, \quad \Delta\left(P_x\right)_{\min} = 0, \quad \Delta\left(P_y\right)_{\min} = \hbar\sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2}, \tag{9}$$

where $\langle . \rangle_\rho$ denotes the inner product on a Hilbert space with metric ρ in which the operators X, Y, P_x and P_y are Hermitian. So far no representation for the two dimensional algebra (8) was provided. Here we find that it can be represented by

$$X = x_0 + \hat{\tau}y_0^2x_0, \quad Y = y_0, \quad P_x = p_{x_0}, \quad \text{and} \quad P_y = p_{y_0} - \hat{\tau}\frac{\hbar}{\theta}y_0^2x_0, \tag{10}$$

where the $x_0, y_0, p_{x_0}, p_{y_0}$ satisfy the common commutation relations for the flat noncommutative space

$$\begin{array}{lll} [x_0,y_0] = i\theta, & [x_0,p_{x_0}] = i\hbar, & [x_0,p_{y_0}] = 0, \\ [p_{x_0},p_{y_0}] = 0, & [y_0,p_{y_0}] = i\hbar, & [y_0,p_{x_0}] = 0, \end{array} \quad \text{for } \theta \in \mathbb{R}. \tag{11}$$

Clearly there exist many more solutions one may construct in this systematic manner from the Ansatz (2)-(7), which will not be our concern here. Instead we will study a concrete model, i.e. the two-dimensional harmonic oscillator on the noncommutative space described by the algebra (8). Using the representation (10), the corresponding Hamiltonian reads

$$\begin{aligned} H_{ncho}^{2D} &= \frac{1}{2m}(P_x^2 + P_y^2) + \frac{m\omega^2}{2}(X^2+Y^2) \\ &= H_{fncho}^{2D} + \frac{\hat{\tau}}{2}\left[m\omega^2\{y_0^2x_0, x_0\} - \frac{\hbar}{m\theta}\{y_0^2x_0, p_{y_0}\}\right] + \frac{\hat{\tau}^2}{2}\left[m\omega^2 + \frac{\hbar^2}{m\theta^2}\right] y_0^2x_0y_0^2x_0 \end{aligned} \tag{12}$$

where we used the standard notation for the anti-commutator $\{A, B\} := AB + BA$. Evidently this Hamiltonian is non-Hermitian with regard to the standard inner product, but respects an antilinear symmetry $\mathcal{PT}_\pm$: $x_0 \to \pm x_0$, $y_0 \to \mp y_0$, $p_{x_0} \to \mp p_{x_0}$, $p_{y_0} \to \pm p_{y_0}$ and $i \to -i$. This suggests that its eigenvalue spectrum might be real, or at least real in parts [10, 11, 12]. Let us now investigate the spectrum perturbatively around the solution of the standard harmonic oscillator. In order to perform such a computation we need to convert flat noncommutative space into the standard canonical variable x_s, y_s, p_{x_s} and p_{y_s}. This is achieved by means of a so-called Bopp-shift $x_0 \to x_s - \frac{\theta}{\hbar}p_{y_s}$, $y_0 \to y_s$, $p_{x_0} \to p_{x_s}$ and

$p_{y_0} \to p_{y_s}$. The Hamiltonian in (12) then acquires the form

$$
\begin{aligned}
H_{ncho}^{2D} &= H_{ho}^{2D} + \frac{m\theta^2\omega^2}{2\hbar^2}p_{y_s}^2 - \frac{m\theta\omega^2}{2\hbar}\{x_s, p_{y_s}\} + \frac{\hat{\tau}}{2}\left[m\omega^2\{y_s^2 x_s, x_s\} - \frac{\hbar}{m\theta}\{y_s^2 x_s, p_{y_s}\}\right] \qquad (13)\\
&\quad + \frac{\hat{\tau}}{2}\left[\left(\frac{1}{m} + \frac{m\theta^2\omega^2}{\hbar^2}\right)\{y_s^2 p_{y_s}, p_{y_s}\} - \frac{m\theta\omega^2}{\hbar}\left(\{y_s^2 p_{y_s}, x_s\} + \{y_s^2 x_s, p_{y_s}\}\right)\right]\\
&\quad - \frac{\hat{\tau}^2}{2}\left[\frac{m\theta\omega^2}{\hbar} + \frac{\hbar}{m\theta}\right]\left(y_s^2 p_{y_s} y_s^2 x_s + y_s^2 x_s y_s^2 p_{y_s}\right) + \frac{\hat{\tau}^2}{2}\left[\frac{1}{m} + \frac{m\theta^2\omega^2}{\hbar^2}\right] y_s^2 p_{y_s} y_s^2 p_{y_s}\\
&\quad + \frac{\hat{\tau}^2}{2}\left[m\omega^2 + \frac{\hbar^2}{m\theta^2}\right] y_s^2 x_s y_s^2 x_s\\
&= H_{ho}^{2D}(x_s, y_s, p_{x_s}, p_{y_s}) + H_{nc}^{2D}(x_s, y_s, p_{x_s}, p_{y_s}).
\end{aligned}
$$

In this formulation we may now proceed to expand perturbatively around the standard two dimensional Fock space harmonic oscillator solution with normalized eigenstates

$$
|n_1 n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}}|00\rangle, \quad a_i^\dagger |n_1 n_2\rangle = \sqrt{n_i + 1}\,|(n_1 + \delta_{i1})(n_2 + \delta_{i2})\rangle, \qquad (14)
$$

$$
a_i |00\rangle = 0, \qquad a_i |n_1 n_2\rangle = \sqrt{n_i}\,|(n_1 - \delta_{i1})(n_2 - \delta_{i2})\rangle, \qquad (15)
$$

for $i = 1, 2$, such that $H_{ho}^{2D}|nl\rangle = E_{nl}^{(0)}|nl\rangle$. The energy eigenvalues for the Hamiltonian H_{ncho}^{2D} then result to

$$
\begin{aligned}
E_{nl}^{(p)} &= E_{nl}^{(0)} + E_{nl}^{(1)} + E_{nl}^{(2)} + \mathcal{O}(\tau^2) \qquad (16)\\
&= E_{nl}^{(0)} + \langle nl| H_{nc}^{2D} |nl\rangle + \sum_{p,q \neq n+l=p+q} \frac{\langle nl| H_{nc}^{2D} |pq\rangle \langle pq| H_{nc}^{2D} |nl\rangle}{E_{nl}^{(0)} - E_{pq}^{(0)}} + \mathcal{O}(\tau^2)\\
&= \omega\hbar(n + l + 1) + \frac{1}{16}\hbar\omega\Omega\left[2n - (2l+1)\Omega + 10l + 6\right]\\
&\quad + \frac{1}{8}\hbar\tau\omega\left[\Omega\left(8nl + 4n + 6l^2 + 10l + 5\right) + 10nl + 5n + 5l^2 + 10l + 5\right] + \mathcal{O}(\tau^2),
\end{aligned}
$$

where $\Omega = m^2\theta^2\omega^2/\hbar^2$. We notice the minus sign in one of the terms, which might be an indication for the existence of an exceptional point [13, 14] in the spectrum. Naturally it would be very interesting to obtain a more precise expression for the eigenenergies, but nonetheless the first order approximations will be very useful for the computation of coherent states [15].

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